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Solving finite boundaries integral equations related to x-ray scanners

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Abstract. Using Fourier transforms within two-dimensional spaces is one of the usual methods employed to obtain a convenient image reconstruction from a sample of transaxial tomography data. However, this task can also be achieved via a thorough study of a Fredholm equation of the first kind with finite boundaries, i.e. admitting a disc as integration domain. With appropriate polar coordinates and utilising furthermore a power series representing $1/r_{MM'}$, the Fredholm equation can be split into a denumerable set of new one-dimensional equations. Afterwards, suitable methods of solution are offered, taking into account the fact that all the equations studied belong to a class of ill-posed problems. The transformations pointed out here will also be helpful for handling as correctly as possible some related and more difficult non-linear problems, e.g. image reconstruction from time-of-flight data measured on ultrasonic pulses.

1. Primary and secondary data

The sampling conditions for the raw numerical data from x-ray scanners are illustrated in figure 1. A source S supplies, through a convenient collimator, a thin parallel beam

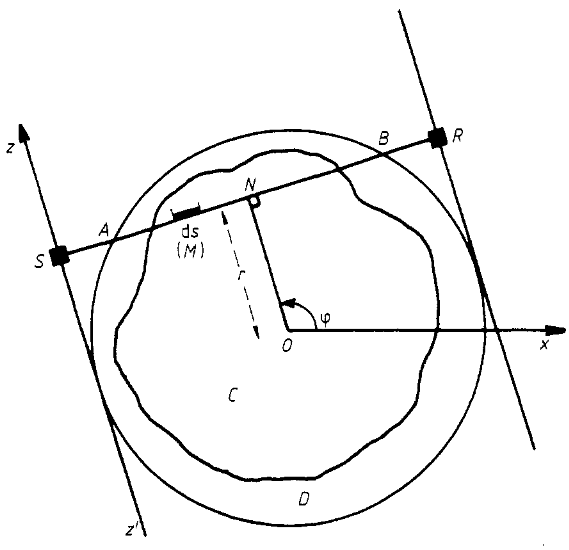

Figure 1. Schematic equipment for primary data measurements.

of x-rays, the intensity of which is measured by the means of an appropriate receiver R. Along the straight path SR, absorption occurs between A and B, i.e. inside the disc D (generally assumed to have unit radius) which contains the studied slice (or virtual cut C) of the investigated object. If I_0 is the initial intensity (at the output S) and I the attenuated intensity at the receiver, we know that

$$I = I_0 \exp\left(- \int_A^B u(M)\, \mathrm{d}s \right) \tag{1.1}$$

where $\mathrm{d}s$ is the elementary length along AB and $u(M)$ the absorption coefficient defined in the close vicinity of point M. It appears then valuable to associate to the middle-point N of AB (polar coordinates r, φ) the positive value

$$J(r, \varphi) = \ln\left(\frac{I_0}{I}\right) = \int_A^B u(M)\, \mathrm{d}s. \tag{1.2}$$

Scannographs are moreover equipped with an automatic device for modifying both the distance $r = ON$ and the angle between ON and the x axis; discretised shifts Δr are first obtained when S and R slide, as a whole, along direction $z'z$ and then, after complete runs, other shifts $\Delta\varphi$ are performed through a rotation around the centre 0 of D. We thus collect a sample of the function $J(r, \varphi)$, corresponding to regularly spaced values of r and φ ($0 \leq r \leq 1, 0 \leq \varphi \leq 2\pi$) and this set constitutes the *primary data*.

Now, to set up an image reconstruction of C, we have to compute everywhere $u(M)$ which is the unknown in equation (1.2). As the integration domain (path AB) does not remain the same for each point N, it appears that the available integrals $J(r, \varphi)$ are not of the usual type characterising Fredholm equations. A direct inversion will nevertheless be possible in such cases, using particular properties of Radon transforms in R^2 spaces (Radon 1917, Guy *et al* 1975). It is, however, also possible to obtain from (1.2) another expression, this time of the Fredholm species. Assuming a sufficient accuracy for our available set of primary data $J(N)$, we may compute a new function $f(M)$, defined as (Barrett and Swindell 1977)

$$f(M) = \oint_\Gamma J(N)\, \mathrm{d}\alpha \tag{1.3}$$

where Γ is the circle admitting OM as diameter while α is the angle OMN (see figure 2). Here, using interpolation methods when necessary, we are able to determine,

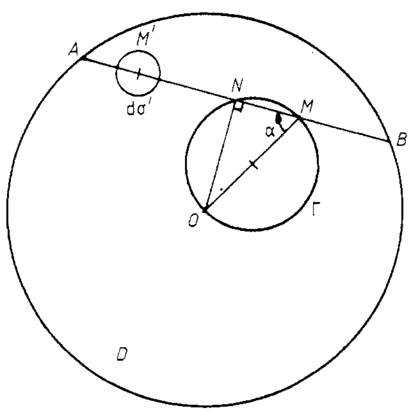

Figure 2. Secondary data computation.

without too much difficulty, any desired value of $f(M)$ through a numerical integration. In practice, we are going to choose once more a discrete and convenient set of points M, in order to attain finally a computerised sample of the *secondary data* $f(M)$.

2. Establishing two types of Fredholm equations†

When all the sampled points M are chosen inside the disc D, we can rewrite formula (1.3) as

$$f(M)=\int\int_D \frac{u(M')}{r_{MM'}}\,\mathrm{d}\sigma'=(\mathbb{K}u)(M). \tag{2.1}$$

To verify the correctness of (2.1), let us examine figure 2 a little further, considering now M as the origin for the polar coordinates $r_{MM'}$ and α, M' representing any point belonging to AB. First, we can split (1.2) into two parts, i.e.

$$J(N)=\int_M^A u(M')\,\mathrm{d}r_{MM'}+\int_M^B u(M')\,\mathrm{d}r_{MM'}. \tag{2.2}$$

During this step, M' goes from M to A for the first integral and from M to B for the second one. Afterwards, we have

$$\mathrm{d}r_{MM'}\,\mathrm{d}\alpha=\frac{\mathrm{d}\sigma'}{r_{MM'}} \tag{2.3}$$

where $\mathrm{d}\sigma'$ symbolises the elementary surface around M'. We find consequently from (1.3)

$$\begin{aligned} f(M)&=\int_0^\pi \mathrm{d}\alpha\left(\int_M^A u(M')\,\mathrm{d}r_{MM'}+\int_M^B u(M')\,\mathrm{d}r_{MM'}\right)\\ &=\int_0^{2\pi}\mathrm{d}\alpha\int_M^A u(M')\,\mathrm{d}r_{MM'}=\int\int_D \frac{u(M')}{r_{MM'}}\,\mathrm{d}\sigma'. \end{aligned} \tag{2.4}$$

The equation obtained is a Fredholm one inside R^2 space: the unchanging integration domain appears to be the whole disc D and we are working with the symmetric kernel

$$K(M,M')=K(M',M)=1/r_{MM'} \tag{2.5}$$

which presents a weak singularity (in Mikhlin's sense) for $M=M'$ (Mikhlin 1960). Moreover, it is interesting to notice that $K(M,M')$ presents a typical convolution structure as $r_{MM'}$ is the length of the difference between two vectors with the same origin, namely

$$r_{MM'}=|\overrightarrow{OM'}-\overrightarrow{OM}|. \tag{2.6}$$

As regards the last symbolism $(\mathbb{K}u)$ (M) utilised in (2.1), the notation $\mathbb{K}$ simply indicates the integral operator simultaneously related to the kernel (2.5) and to the bounded integration domain D.

† In the present paper, we consider as a Fredholm equation (in a somewhat broad sense) any linear integral equation linked to an invariable integration domain D.

To obtain another Fredholm equation, let us denote by Π the whole plane R^2 containing D. To build up equation (2.1), we have implicitly admitted the constraint $(M, M') \in D \times D$ but this is not absolutely necessary. When we are allowed to consider also points outside $D(\Rightarrow(M, M') \in \Pi \times \Pi)$, it will be convenient to write down the extended transform

$$f(M) = \iint_{\Pi} \frac{u(M')}{r_{MM'}} \, d\sigma' \tag{2.7}$$

which results from remarks analogous to those previously pointed out in (2.2)-(2.4). We have only to start from

$$J(N) = \int_L u(M') \, ds \tag{2.8}$$

L being the whole straight line perpendicular to ON at N. Furthermore, when $M' \notin D$, we will assume $u(M') = 0$ as we want an unchanged function $f(M)$ inside D. Formula (1.3) remains valid for $M \notin D$ and the new geometry for the integration along Γ is clearly illustrated in figure 3.

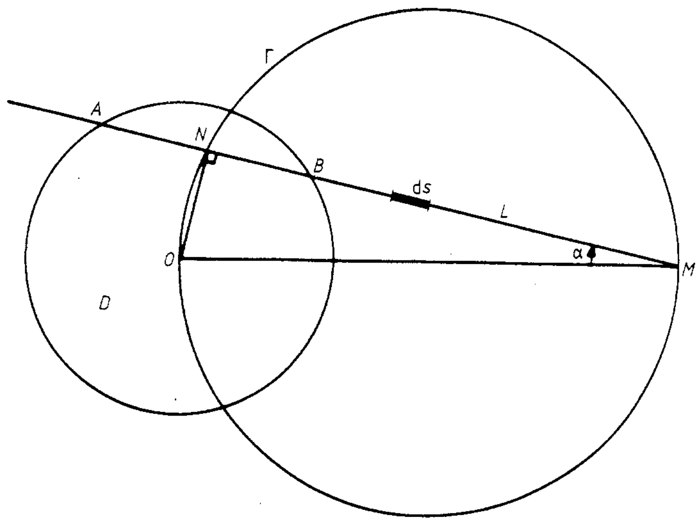

Figure 3. Prolonged computation for $f(M)$ with $u(M') = 0$ if $M' \notin D(\Rightarrow J(N) = 0$ if $N \notin D)$.

Equation (2.7) is very attractive as the convolution structure of the kernel (2.5) is suitably associated with the unboundedness of the integration plane Π. In such favourable circumstances, we know that the convolution theorem can be applied (Papoulis 1962, Brigham 1974). We have consequently

$$\boldsymbol{F}_2(f) = \boldsymbol{F}_2(K) \cdot \boldsymbol{F}_2(u) \tag{2.9}$$

leading to the inversion

$$u = \boldsymbol{F}_2^{-1}[\boldsymbol{F}_2(f)/\boldsymbol{F}_2(K)] \tag{2.10}$$

$\boldsymbol{F}_2$ representing the operator associated with Fourier transforms in R^2 spaces; it appears that (2.10) is a real key formula which is presently widely employed for effective image reconstructions.

3. Splitting the bounded Fredholm equation

We are now going to focus our attention on the bounded Fredholm equation (2.1) (i.e. for which the integral operator is working upon a *bounded* domain, here D). Unfortunately, as the convolution theorem cannot work when we have to deal with finite areas, we must construct other inversion techniques, very different from the satisfactory one corresponding to formula (2.10).

The cylindrical symmetry, which characterises the integration domain D, must firstly be recalled. Then, utilising the polar coordinates (r, θ, φ) and (r', θ', φ') to specify M and M' respectively, and putting

$$\begin{aligned} r_< &= \inf(r, r') \\ r_> &= \sup(r, r') \end{aligned} \tag{3.1}$$

we can represent $1/r_{MM'}$, within R^3 space, by the expansion (Eyring *et al* 1946)

$$\frac{1}{r_{MM'}} = \sum_{n=0}^{\infty} \sum_{m=-n}^{m=+n} \frac{(n-|m|)!}{(n+|m|)!} \frac{r_<^n}{r_>^{n+1}} P_n^{|m|}(\cos\theta) P_n^{|m|}(\cos\theta') \exp[\mathrm{i}m(\varphi-\varphi')]. \tag{3.2}$$

In (3.2), $P_n^{|m|}(\cos\theta)$ denotes the associated Legendre polynomial of argument $\cos\theta$, such that (Eyring *et al* 1946, Magnus *et al* 1966)

$$P_n^{|m|}(x) = \frac{(1-x^2)^{m/2}}{2^n n!} \frac{\mathrm{d}^{n+|m|}}{\mathrm{d}x^{n+|m|}} [(x^2-1)^n]. \tag{3.3}$$

For our present problem, we only need an expansion of $1/r_{MM'}$ inside R^2 space. The required formula is deduced from (3.2) by putting $\theta = \theta' = \pi/2$ ($\Rightarrow \cos\theta = \cos\theta' = 0$). The kernel becomes

$$K(M, M') = \sum_{n=0}^{\infty} \sum_{m=-n}^{m=+n} \frac{(n-|m|)!}{(n+|m|)!} \frac{r_<^n}{r_>^{n+1}} [P_n^{|m|}(0)]^2 \exp[\mathrm{i}m(\varphi-\varphi')]. \tag{3.4}$$

The constant values $P_n^{|m|}(0)$ are deduced from (3.3) and are well known. If we have simultaneously $|m| \leqslant n$ and $|m|+n$ even, then

$$P_n^{|m|}(0) = (-1)^{(n-|m|)/2} \frac{(n+|m|)!}{2^n[(n-|m|)/2]![(n+|m|)/2]!}. \tag{3.5}$$

In all other cases $P_n^{|m|}(0) = 0$.

Replacing now n by $|m|+2l$ in (3.4), we can write down an interesting expansion for $K(M, M')$, namely

$$K(M, M') = \sum_{m=0}^{\infty} k_m(r, r')[\cos(m\varphi)\cos(m\varphi') + \sin(m\varphi)\sin(m\varphi')] \tag{3.6}$$

with the following definitions of the introduced auxiliary kernels

$$\begin{aligned} k_0(r, r') &= \frac{1}{r_>} \sum_{l=0}^{\infty} [P_{2l}^0(0)]^2 z^{2l} \\ k_m(r, r') &= \frac{2z^m}{r_>} \sum_{l=0}^{\infty} \frac{(2l)!}{(2m+2l)!} [P_{m+2l}^m(0)]^2 z^{2l} \qquad (m>0) \end{aligned} \tag{3.7}$$

where z is the ratio $r_</r_>$. As $d\sigma' = r'dr'd\varphi'$, our primary equation (2.1) becomes, utilising (3.7),

$$f(M) = f(r, \varphi) = \sum_{m=0}^{\infty} \int_0^1 r' k_m(r, r')\, dr' \times \int_0^{2\pi} [\cos(m\varphi)\cos(m\varphi') + \sin(m\varphi)\sin(m\varphi')] u(r', \varphi') d\varphi'. \tag{3.8}$$

At the present stage, it seems highly convenient to write down the Fourier series (as regards φ or φ') associated with both bounded functions $f(r, \varphi)$ and $u(r', \varphi')$, such that

$$\begin{aligned} f(r, \varphi) &= f_0(r) + \sum_{m=1}^{\infty} [f_{c,m}(r)\cos(m\varphi) + f_{s,m}(r)\sin(m\varphi)] \\ u(r', \varphi') &= \frac{u_0(r')}{2\pi} + \frac{1}{\pi}\sum_{m=1}^{\infty} [u_{c,m}(r')\cos(m\varphi') + u_{s,m}(r')\sin(m\varphi')] \end{aligned} \tag{3.9}$$

where indices c and s used in partial functions f and u merely indicate that we have a cosine or a sine multiplication factor. Putting expansions (3.9) into (3.8) and then integrating over φ' from 0 to 2π gives straightforwardly a splitting of our initial equation (2.1). We now have to solve the following unidimensional equations:

$$f_0(r) = \frac{1}{2\pi}\int_0^{2\pi} f(r, \varphi)\, d\varphi = \int_0^1 r' k_0(r, r') u_0(r')\, dr'$$

and for $m > 0$:

$$\begin{aligned} f_{c,m}(r) &= \frac{1}{\pi}\int_0^{2\pi} f(r, \varphi)\cos(m\varphi)\, d\varphi = \int_0^1 r' k_m(r, r') u_{c,m}(r')\, dr' \\ f_{s,m}(r) &= \frac{1}{\pi}\int_0^{2\pi} f(r, \varphi)\sin(m\varphi)\, d\varphi = \int_0^1 r' k_m(r, r') u_{s,m}(r')\, dr'. \end{aligned} \tag{3.10}$$

We are thus given 'infinitely many' integral equations of the first kind and we have easy access to all inhomogeneous terms $f_0(r), f_{c,m}(r)$ and $f_{s,m}(r)$; the kernels $r'k_m(r, r')$ are also completely known through formulae (3.7). If we are able to solve these uncoupled equations, the determination of $u(r, \varphi)$ will result from a mere sum as indicated in the second formula (3.9).

4. Solving possibilities for ill-posed integral equations

Many authors have emphasised the fact that Fredholm equations of the first kind belong to a class of ill-posed problems. Starting from a function $u(x)$, we can build up $f(x)$ through the transformation $f(x) = (\mathbb{K}u)(x)$, $\mathbb{K}$ being a Fredholm integral operator. Conversely, when the preceding function $f(x)$ is given, we know that $u(x)$ is effectively a solution for $(\mathbb{K}u)(x) = f(x)$: however, this last equation will be said to be an ill-posed one if some slight modifications performed upon $f(x)$ (at least, round-off or random errors in measurements) are able to induce unbounded changes in $u(x)$ (Hadamard 1902).

It is not very difficult to verify that equation (2.4) is effectively ill-posed. A quick check of this results from an examination of the transforms of two very simple functions

$u(M')$, both presenting a cylindrical symmetry around O and consequently noted $u(r')$. On the one hand, when we choose $u_1(r') \equiv 1$, we find that (see appendix and figure 4)

$$f_1(r) = \iint_D \frac{d\sigma'}{r_{MM'}} = 4E(r) \tag{4.1}$$

where $E(r)$ is the complete elliptic integral of the second kind (Jahnke *et al* 1960, Magnus *et al* 1966), i.e.

$$E(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 \chi)^{1/2} \, d\chi$$

$$\Rightarrow \forall r \in [0, 1] \qquad 4 \leq f_1(r) \leq 2\pi. \tag{4.2}$$

On the other hand, starting from $u_2(r') = 1/(1 - r'^2)^{1/2}$, we obtain the result (see the appendix)

$$f_2(r) = \iint_D \frac{d\sigma'}{r_{MM'}(1 - r'^2)^{1/2}} = \pi^2. \tag{4.3}$$

Here, $u_2(r')$ becomes unbounded for $r' = 1$ but the integral (3.8) remains computable. Now, if $f_1(r) = 4E(r)$ is slightly modified and replaced by

$$f(r) = 4E(r) + \varepsilon \tag{4.4}$$

we can tell from (4.1) and (4.3) that an original is

$$u(r') = 1 + \frac{\varepsilon}{\pi^2} \frac{1}{(1 - r'^2)^{1/2}}. \tag{4.5}$$

Thus, a weak distortion of $f_1(r)$ (addition of a small value ε) is clearly able to lead to a noticeable unboundedness of $u(r')$ as soon as r' is sufficiently close to one.

Obviously, when we wish to reconstruct an original from an enciphered transform, it becomes necessary to avoid such troubles. An interesting technique of assays (Tikhonov and Arsenine 1976) consists of working according to the following lines.

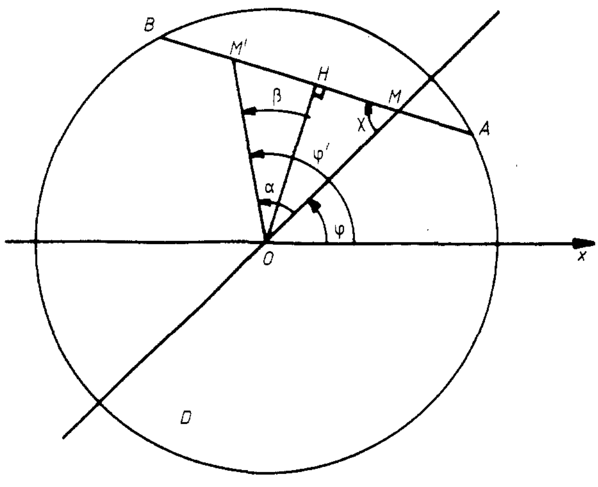

Figure 4. Parameters and variables for calculation of integrals: $|\overline{OM}| = r$; $|\overline{OM'}| = r'$; distance $HM' = s$; and angles α, β, φ, φ' and χ are as shown.

Firstly, we will choose a set of functions $v_0(M'), v_1(M'), \ldots, v_n(M'), \ldots$, such that any acceptable solution $u(M')$ can be represented as a sum

$$u(M') = \sum_{k=0}^{\infty} a_k v_k(M'). \tag{4.6}$$

Then, we have to calculate all the direct transforms

$$f_k(M) = \iint_D \frac{v_k(M')\,\mathrm{d}\sigma'}{r_{MM'}}. \tag{4.7}$$

Next, $f(M)$ being given, we have to find the best approximate solution (e.g. through the least-squares method) for the finite expression

$$f(M) \approx \sum_{k=0}^{n} a_k f_k(M). \tag{4.8}$$

A sufficiently accurate knowledge of the coefficients a_k present in (4.8) immediately gives an approximate original $u(M')$, without theoretical defects, equal to the sum (4.6) truncated just after the term $a_n v_n(M')$; such a process automatically eliminates from $f(M)$ the unwanted parts which lead to a poor and frequently false reconstruction of $u(M')$.

5. Computing properties of some simple functions

Let us return to the uncoupled equations (3.10). Some further remarks are very helpful here when we try to solve this system if we assume that the whole solution $u(r', \varphi')$ shall be analytic (Whittaker and Watson 1927, Flanigan 1972) (all partial functions $u_{c,m}(r')\cos(m\varphi')$ and $u_{s,m}(r')\sin(m\varphi')$ are then also analytic). Consequently, power expansions of $u_{c,m}(r')$ (or of $u_{s,m}(r')$) will present the form

$$u_{c,m}(r') = r'^m(c_0 + c_2 r'^2 + \ldots + c_{2k} r'^{2k} + \ldots). \tag{5.1}$$

Series of the preceding type are actually necessary as any term $r'^p \cos(m\varphi')$ will be analytic if and only if $p = m + 2k$ $(k \in \mathbb{N})$.

Now, it becomes interesting to equate each $u_{c,m}(r')$ (or $u_{s,m}(r')$) to a sum closely related to (4.6). Let us write, recalling in our formalism the existence of parameter m,

$$\begin{aligned} u_{c,m}(r') &= \sum_{k=0}^{\infty} a_{c,k}(m) w_k(r'; m) \\ u_{s,m}(r') &= \sum_{k=0}^{\infty} a_{s,k}(m) w_k(r'; m). \end{aligned} \tag{5.2}$$

Here, for a given m, we must use a set $w_k(r'; m)$ forming a kind of basis such that linear combinations of the w_k are able to give all the authorised series (5.1) and nothing else. Note that the w_k are independent of the angle φ' while this is not usually the case for the v_k present in (4.6).

An important choice of functions $w_k(r'; m)$ is obviously available but, in practice, it appears most convenient to select those for which the transforms (4.7) will be easy enough to compute. In that sense, the following set:

$$w_k(r'; m) = r'^m(1 - r'^2)^k \tag{5.3}$$

seems attractive as the related transforms

$$f_k(r;\, m)\cos(m\varphi) = \iint_D \frac{r'^m(1-r'^2)^k}{r_{MM'}}\cos(m\varphi')\,\mathrm{d}\sigma' \tag{5.4}$$

are such that the radial functions $f_k(r;\, m)$ can be computed through finite sums of unidimensional integrals. The results are (see the appendix)

$$\begin{aligned} f_k(r;\, m) &= (-1)^{m/2}\sum_{p=0}^{m/2} T_{mpk}r^{m-2p} \\ &\quad\times\int_0^{\pi/2}\cos(m\chi)\sin^{m-2p}\chi(1-r^2\sin^2\chi)^{p+k+1/2}\,\mathrm{d}\chi \qquad \text{for } m \text{ even} \\ f_k(r;\, m) &= (-1)^{(m-1)/2}\sum_{p=0}^{(m-1)/2} T_{mpk}\, r^{m-2p} \\ &\quad\times\int_0^{\pi/2}\sin(m\chi)\sin^{m-2p}\chi(1-r^2\sin^2\chi)^{p+k+1/2}\,\mathrm{d}\chi \qquad \text{for } m \text{ odd} \end{aligned} \tag{5.5}$$

with the parametric values

$$\begin{aligned} T_{mpk} &= 4(-1)^p\binom{m}{2p}\int_0^1 t^{2p}(1-t^2)^k\,\mathrm{d}t \\ &= 2(-1)^p k!\binom{m}{2p}\left(\prod_{j=p}^{p+k}(j+\tfrac{1}{2})\right)^{-1}. \end{aligned} \tag{5.6}$$

6. Solved examples

To show clearly how to use formulae (5.1)-(5.5), we give two examples of a $u(r',\varphi')$ reconstruction. A careful prior determination of the transform

$$f(r) = \iint_D \frac{\exp(-r'^2)}{r_{MM'}}\,\mathrm{d}\sigma' \tag{6.1}$$

gave the sampled values gathered in table 1.

For $m=0$, our corresponding basic functions are

$$\begin{aligned} &w_k(r';\, 0) = (1-r'^2)^k \\ &\Rightarrow f_k(r;\, 0) = \frac{2^{2k+2}(k!)^2}{(2k+1)!}\int_0^{\pi/2}(1-r^2\sin^2\chi)^{k+1/2}\,\mathrm{d}\chi. \end{aligned} \tag{6.2}$$

Table 1.

r	$f(r)$	r	$f(r)$
0.0	4.692 435	0.6	3.753 514
0.1	4.663 292	0.7	3.461 057
0.2	4.577 032	0.8	3.145 942
0.3	4.437 042	0.9	2.807 128
0.4	4.248 597	1.0	2.398 127
0.5	4.018 333		

Utilising (6.2) to determine numerically the first four functions $f_k(r; 0)$ $(0 \leqslant k \leqslant 3)$, we obtained by the least-squares method

$$f(r) \simeq 0.365\,749 f_0(r; 0) + 0.389\,622 f_1(r; 0) + 0.124\,387 f_2(r; 0) + 0.120\,298 f_3(r; 0)$$
$$\Rightarrow u(r') \simeq 1.000\,056 - 0.999\,289 r'^2 + 0.485\,280 r'^4 - 0.120\,298 r'^6. \tag{6.3}$$

The last obtained function $u(r')$ appears to be a rather good reconstruction of the original, especially when we remember that only four basic functions $w_k(r'; 0)$ were effectively employed. A quick comparison between true and reconstructed values is permitted through table 2.

For our second example, we decided to start from the original function

$$u(r', \varphi') = \frac{1}{\pi} r'^2(a_0 + a_2 r'^2 + a_4 r'^4) \cos(2\varphi')$$
$$= \frac{1}{\pi} u_{c,2}(r') \cos(2\varphi') \tag{6.4}$$

the coefficients of the radial part being chosen in such a way that

$$\left.\begin{aligned} &u_{c,2}(1) = 0.9 \\ &u_{c,2}(0.7) = 1.5 \\ &[\mathrm{d}u_{c,2}/\mathrm{d}r']_{r'=0.7} = 0 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} &a_0 = 7.068\,585 \\ &a_2 = -10.109\,177 \\ &a_4 = 3.940\,592. \end{aligned}\right. \tag{6.5}$$

We have firstly computed a sample of values for the transform $f_{c,2}(r)$ (i.e. when $r = 0.00, 0.05, 0.10, \ldots, 1.00$), utilising the integrals (3.10). Moreover, a random Gaussian error (with a standard deviation $\sigma = 0.001$) was added to each result, in order to examine empirically the image reconstruction behaviour under the effects of such randomised discrepancies. Some of the so-obtained ciphered data are now given in table 3.

Table 2.

r'	$\exp(-r'^2)$	$u(r')$ reconstructed	Absolute relative error
0.0	1.000 00	1.000 06	6×10^{-5}
0.2	0.960 79	0.960 85	7×10^{-5}
0.4	0.852 14	0.852 10	5×10^{-5}
0.6	0.697 68	0.697 59	1×10^{-4}
0.8	0.527 29	0.527 75	9×10^{-4}
1.0	0.367 88	0.365 75	6×10^{-3}

Table 3.

r	$f_{c,2}(r)$	r	$f_{c,2}(r)$
0.0	0.000 00	0.6	1.974 25
0.1	0.080 45	0.7	2.314 80
0.2	0.311 78	0.8	2.538 04
0.3	0.665 12	0.9	2.631 53
0.4	1.096 29	1.0	2.608 41
0.5	1.551 41		

The next step was a careful determination of the first four transforms $f_k(r; 2)$ corresponding to $k = 0, 1, 2$ and 3, respectively, related to $w_k(r'; 2) = r'^2(1-r'^2)^k$. Using once more the least-squares method, we found

$$f_{c,2}(r) \approx 0.868\,862 f_0(r; 2) + 2.255\,746 f_1(r, 2) + 3.969\,987 f_2(r; 2) - 0.030\,124 f_3(r; 2) \quad (6.6)$$

instead of the true representation given by

$$f_{c,2}(r) = 0.9 f_0(r; 2) + 2.2280 f_1(r; 2) + 3.9406 f_2(r; 2). \quad (6.7)$$

It appears from (6.6) that our u function will be reconstructed through

$$u_{c,2}(r') \simeq r'^2[0.868\,862 + 2.255\,746(1-r'^2) + 3.969\,987(1-r'^2)^2 - 0.030\,124(1-r'^2)^3] \quad (6.8)$$

and we give in table 4 an interesting comparison of both functions $u_{c,2}(r')$ (true and reconstructed, respectively denoted u_T and u_R). Let us also indicate that the same technique, when applied to the accurate function $f_{c,2}(r)$ (i.e. without random errors), gives a reconstructed $u_{c,2}(r')$ which does not differ from u_T by more than 7.0×10^{-7} everywhere on [0, 1].

Table 4.

r'	u_T	u_R	r'	u_T	u_R
0.0	0.0000	0.0000	0.6	1.4184	1.4151
0.1	0.0697	0.0696	0.7	1.5000	1.4935
0.2	0.2668	0.2667	0.8	1.4162	1.4042
0.3	0.5572	0.5568	0.9	1.1871	1.1669
0.4	0.8883	0.8875	1.0	0.9000	0.8689
0.5	1.1969	1.1953			

7. Future schemes and conclusion

We have described here a theoretical method for solving the integral equation (2.1) which is related to x-ray scanners and corresponds to a bounded integration domain (disc D). Such a singular problem of the first kind was not previously studied in such detail because equation (2.7) is frequently also available and leads to an image reconstruction that is much easier to attain. In fact, the present computing techniques are not actually efficient enough to challenge the other well known methods of treating x-ray data to obtain accurate image reconstructions. Their main interest lies elsewhere as they are linked to a different mathematical problem involving as a prime constraint the boundedness of the integration disc D; they tell us how to proceed and reach the required solution when the secondary data $f(M)$ become impossible to determine (or even to define) outside D.

For instance, such a lack of information is patent for $f(M)$ when we are given as primary data times-of-flight of ultrasonic pulses through a studied object. At first sight, $u(M)$ now representing the inverse of the ultrasonic velocity instead of the absorption coefficient, this new problem seems very similar to the previous one. We can write once more, exactly as in (1.2),

$$J(N)=\int_{\widehat{AB}} u(M)\,\mathrm{d}s \tag{7.1}$$

where N is the midpoint between A (input) and B (output). The trouble is, however, that the paths followed by phonons are not straight. Furthermore, the problem cannot be limited to a plane and this is another difficulty. Without entering into more details, let us indicate that transforms similar to (1.3) can be achieved, leading to a new kind of secondary data $f(M)$. This time, the correspondence between $u(M)$ and $f(M)$ is no longer linear and it is easily understandable that $f(M)$ will be rather difficult to invert (Guy 1986). When there is not too great a distortion from linearity, it was, however, proposed (McKinnon and Bates 1980) to solve first an approximate equation analogous to (2.1), suitable iterative corrections being tried afterwards. Unfortunately, we have to remember that we cannot adjoin to (2.1) any unbounded equation resembling (2.7) as $f(M)$ is not defined here for $M \notin D$ ($\Rightarrow$formula (2.10) does not work). Finally, for such problems of ultrasonic imaging, it is clear that we have to know how to solve (2.1) directly and the present paper contributes an answer to that important question.

Appendix. Practical computation of some transforms

To work out formulae (5.4), it appears necessary to calculate without too many difficulties the transforms related to the two following types of functions:

$$\begin{aligned} U_{\mathrm{c}}(r',\varphi') &= r'^m\cos(m\varphi')g(r'^2)\\ U_{\mathrm{s}}(r',\varphi') &= r'^m\sin(m\varphi')g(r'^2). \end{aligned} \tag{A1}$$

In complex notation, both cases can be treated simultaneously. Defining $U(r',\varphi')$ through

$$U(r',\varphi')=U_{\mathrm{c}}(r',\varphi')+\mathrm{i}U_{\mathrm{s}}(r',\varphi')=r'^m\exp(\mathrm{i}m\varphi')g(r'^2) \tag{A2}$$

we have, since $\varphi'=\varphi+\alpha$ (see figure 4),

$$\begin{aligned} F(r,\varphi) &= F_{\mathrm{c}}(r,\varphi)+\mathrm{i}F_{\mathrm{s}}(r,\varphi)\\ &= \iint_D \frac{U(r',\varphi')}{r_{MM'}}\,\mathrm{d}\sigma'\\ &= \exp(\mathrm{i}m\varphi)\int_{-\pi/2}^{+\pi/2}\mathrm{d}\chi\int_{-[1-r^2\sin^2\chi]^{1/2}}^{[1-r^2\sin^2\chi]^{1/2}}\exp(\mathrm{i}m\alpha)r'^m g(r'^2)\,\mathrm{d}s\\ &= 2\exp(\mathrm{i}m\varphi)\int_0^{\pi/2}\mathrm{d}\chi\int_{-[1-r^2\sin^2\chi]^{1/2}}^{[1-r^2\sin^2\chi]^{1/2}}\cos(m\alpha)r'^m g(r'^2)\,\mathrm{d}s. \end{aligned} \tag{A3}$$

To understand the last transformation performed, it is sufficient to notice that for a change of sign of χ (symmetry as regards OM), $\cos(m\alpha)$ will remain unchanged while there is also a change of sign for $\sin(m\alpha)$.

Afterwards, taking into account (see figure 4)

$$\alpha=\pi/2+(\beta-\chi) \tag{A4}$$

we are able to write, using $F(r, \varphi) = 2\exp(\mathrm{i}m\varphi)G(r; m)$,

$$G(r; m) = \mathrm{Re}\left(\int_0^{\pi/2} \mathrm{d}\chi \int_{-[1-r^2\sin^2\chi]^{1/2}}^{[1-r^2\sin^2\chi]^{1/2}} \exp(\mathrm{i}m\alpha) r'^m g(r'^2)\, \mathrm{d}s\right)$$

$$= \mathrm{Re}\left(\mathrm{i}^m \int_0^{\pi/2} \exp(-\mathrm{i}m\chi)\, \mathrm{d}\chi \int_{-[1-r^2\sin^2\chi]^{1/2}}^{[1-r^2\sin^2\chi]^{1/2}} \exp(\mathrm{i}m\beta) r'^m g(r'^2)\, \mathrm{d}s\right). \quad \text{(A5)}$$

Now, for two points M_1' and M_2' placed on AB and equidistant from H (i.e. symmetric as regards OH), we have the same value for $\cos(m\beta)$ and a change of sign for $\sin(m\beta)$. This remark leads to

$$G(r; m) = 2\mathrm{Re}\left(\mathrm{i}^m \int_0^{\pi/2} \exp(-\mathrm{i}m\chi)\, \mathrm{d}\chi \int_0^{[1-r^2\sin^2\chi]^{1/2}} \cos(m\beta) r'^m g(r'^2)\, \mathrm{d}s\right). \quad \text{(A6)}$$

Considering furthermore that (see figure 4)

$$r'\cos\beta = r\sin\chi \qquad r'\sin\beta = s \quad \text{(A7)}$$

we obtain

$$r'^m\cos(m\beta) = \mathrm{Re}[(r\sin\chi + \mathrm{i}s)^m]$$

$$= \sum_{p=0}^{N} (-1)^p \binom{m}{2p} s^{2p} r^{m-2p} \sin^{m-2p}\chi \quad \text{(A8)}$$

where N is the integer equal to or immediately preceding $m/2$ (i.e. $N = m/2$ for m even and $N = (m-1)/2$ for m odd). Utilising (A8), it becomes possible to write more explicitly two formulae giving $G(r; m)$. With the new variable

$$z = s/[1 - r^2\sin^2\chi]^{1/2} \quad \text{(A9)}$$

the results are

$$G(r; m) = 2\sum_{p=0}^{m/2} (-1)^{m/2+p} \binom{m}{2p} r^{m-2p} \int_0^{\pi/2} \cos(m\chi) \sin^{m-2p}\chi (1 - r^2\sin^2\chi)^{p+1/2}\, \mathrm{d}\chi$$

$$\times \int_0^1 z^{2p} g[z^2 + r^2\sin^2\chi(1-z^2)]\, \mathrm{d}z \qquad \text{for } m \text{ even} \quad \text{(A10)}$$

and

$$G(r; m) = 2\sum_{p=0}^{(m-1)/2} (-1)^{(m-1)/2+p} \binom{m}{2p} r^{m-2p}$$

$$\times \int_0^{\pi/2} \sin(m\chi) \sin^{m-2p}\chi (1 - r^2\sin^2\chi)^{p+1/2}\, \mathrm{d}\chi$$

$$\times \int_0^1 z^{2p} g[z^2 + r^2\sin^2\chi(1-z^2)]\, \mathrm{d}z \qquad \text{for } m \text{ odd.} \quad \text{(A11)}$$

Finally, when we choose

$$g(r'^2) = (1 - r'^2)^k \quad \text{(A12)}$$

formulae (A10) and (A11) are changed into formulae (5.4) given previously.

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